

Korovkin type theorem for iterates of certain positive linear operators

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Submitted: March 15, 2011

Abstract

In this paper we prove that if $T : C[0, 1] \rightarrow C[0, 1]$ is a positive linear operator with $T(e_0) = 1$ and $T(e_1) - e_1$ does not change the sign, then the iterates T^m converges to some positive linear operator $T^\infty : C[0, 1] \rightarrow C[0, 1]$ and we derive quantitative estimates in terms of moduli of smoothness. This result enlarges the class of operators for which the limit of the iterates can be computed and the quantitative estimates of iterates can be given.

1 Introduction and Main results

The methods employed to study the convergence of iterates of some operators include Matrix Theory methods, like stochastic matrices, Korovkin-type theorems, quantitative results about the approximation of functions by positive linear operators, point theorems, or methods from the theory of C_0 -semigroups, like Trotter's approximation theorem, see [1]-[19]. However, these results fail to calculate the iterate limit and give the quantitative estimates of iterates for many classical operators. In respect to this, we mention recent works by I. Gavrea and M. Ivan [5], [6] and I. Rasa [15], [16].

In this paper we establish quantitative Korovkin type theorem for the iterates of certain positive linear operators $T : C[0, 1] \rightarrow C[0, 1]$ satisfying $T(e_0) = e_0$, $T(e_1) - e_1 \leq 0$ (or ≥ 0). As a consequence of our results, we obtain the quantitative estimates for the iterates of almost all classical and new positive linear operators. Notice that quantitative Korovkin type theorems for a sequence of positive linear operators are studied in [20], [21].

Let $C[0, 1]$ be the set of all real-valued and continuous functions defined on the compact interval $[0, 1]$ endowed with the sup-norm $\|f\| := \sup \{|f(x)| : x \in [0, 1]\}$. $W_{2,\infty}[0, 1]$ is defined as follows

$$W_{2,\infty}[0, 1] := \{f \in C[0, 1] : f' \text{ absolutely continuous, } \|f''\|_{L_\infty} < \infty\},$$

$$\|f\|_{L_\infty} := \text{vrai max } \{|f''(x)| : 0 \leq x \leq 1\},$$

and L_∞ is the space of essentially bounded measurable functions endowed with $\|\cdot\|_{L_\infty}$ norm.

The main tools to measure the degree of convergence of the powers of positive linear operators are the moduli of smoothness of first and second order. For $f \in C[0, 1]$ and $\delta \geq 0$ we have

$$\omega_1(f; \delta) := \sup \{|f(x+h) - f(x)| : x, x+h \in [0, 1], 0 \leq h \leq \delta\},$$

$$\omega_2(f; \delta) := \sup \{|f(x+h) - 2f(x) + f(x-h)| : x, x \pm h \in [0, 1], 0 \leq h \leq \delta\}.$$

For $f \in C[0, 1]$ we define the extension $f_h : [-h, 1+h] \rightarrow \mathbb{R}$, with $h > 0$, by

$$f_h(x) := \begin{cases} P_-(x), & -h \leq x \leq 0, \\ f(x), & 0 \leq x \leq 1, \\ P_+(x), & 1 \leq x \leq 1+h, \end{cases}$$

where P_- and P_+ are at most order best approximants to f on the indicated intervals. Then Zhuk's function $Z_h f$ is defined by means of the second order Steklov means

$$Z_h f(x) := \frac{1}{h} \int_{-h}^h \left(1 - \frac{|t|}{h}\right) f_h(x+t) dt, \quad 0 \leq x \leq 1.$$

It can be shown that $Z_h f \in W_{2,\infty}[0,1]$. It is known that for sufficiently large l and a fixed $\varepsilon > 0$ we have

$$\begin{aligned} \|f - g\| &\leq \|f - Z_h f\| + \|B_l(Z_h f) - Z_h f\| \leq \frac{3}{4}\omega_2(f; h) + \varepsilon, \\ \|g'\| &\leq \|(Z_h f)'\| \leq \frac{1}{h} \left(2\omega_1(f; h) + \frac{3}{2}\omega_2(f; h)\right), \\ \|g''\| &\leq \|(Z_h f)''\|_{L_\infty} \leq \frac{1}{\delta^2} \frac{3}{2}\omega_2(f; \delta). \end{aligned} \quad (1)$$

For any positive linear operator $T : C[0,1] \rightarrow C[0,1]$, we define the powers of T by

$$T^0 = I, \quad T^1 = T, \quad T^{m+1} = T \circ T^m, \quad m \in \mathbb{N}.$$

Let $e_i : [0,1] \rightarrow \mathbb{R}$ be the monomial functions $e_i(x) = x^i$, $i = 0, 1, 2$.

Now we formulate the main results of the paper. It shows that under the conditions $T(e_0) = 1$ and $T(e_1) - e_1$ does not change the sign, the iterates of $T : C[0,1] \rightarrow C[0,1]$ converges to some linear positive operator $T^\infty : C[0,1] \rightarrow C[0,1]$.

Theorem 1 *Suppose that $T : C[0,1] \rightarrow C[0,1]$ is a positive linear operator such that $T(e_0) = e_0$.*

(a) *If $T(e_1) \leq e_1$, then there exists a linear positive operator $T^\infty : C[0,1] \rightarrow C[0,1]$ such that*

$$\lim_{n \rightarrow \infty} T^n(f) = T^\infty(f), \quad f \in C[0,1],$$

and the following pointwise estimate

$$\begin{aligned} |T^\infty(f; x) - T^m(f; x)| &\leq 3\omega_2(f; |(T^m - T^\infty)(e_1; x)|) + 2\omega_1(f; |(T^m - T^\infty)(e_1; x)|) \\ &\quad + \frac{3}{4}\omega_2\left(f; \sqrt{|(T^\infty - T^m)(e_2; x)|} + 2|(T^m - T^\infty)(e_1; x)|\right) \end{aligned} \quad (2)$$

holds true for $x \in [0,1]$ and $f \in C[0,1]$.

(b) *If $T(e_1) \geq e_1$, then there exists a linear positive operator $T^\infty : C[0,1] \rightarrow C[0,1]$ such that*

$$\lim_{n \rightarrow \infty} T^n(f) = T^\infty(f), \quad f \in C[0,1],$$

and the following pointwise estimate

$$\begin{aligned} |T^\infty(f; x) - T^m(f; x)| &\leq 3\omega_2(f; |(T^m - T^\infty)(e_1; x)|) + 2\omega_1(f; |(T^m - T^\infty)(e_1; x)|) \\ &\quad + \frac{3}{4}\omega_2\left(f; \sqrt{|(T^\infty - T^m)(e_2; x)|}\right) \end{aligned} \quad (3)$$

holds true for $x \in [0,1]$ and $f \in C[0,1]$.

Corollary 2 *Suppose that $T : C[0,1] \rightarrow C[0,1]$ is a positive linear operator such that*

$$T(e_0) = e_0, \quad T(e_1) = e_1. \quad (4)$$

Then there exists a linear positive operator $T^\infty : C[0,1] \rightarrow C[0,1]$ such that

$$\lim_{m \rightarrow \infty} \|T^\infty(f) - T^m(f)\| = 0, \quad f \in C[0,1].$$

Furthermore the following pointwise estimate

$$|T^\infty(f; x) - T^m(f; x)| \leq \frac{3}{4}\omega_2\left(f; \sqrt{|(T^\infty - T^m)(e_2; x)|}\right) \quad (5)$$

holds true for $x \in [0,1]$ and $f \in C[0,1]$.

The second result shows that under the conditons $T(e_0) = e_0$, $T(e_1) = e_1$, $T^\infty(e_2) = e_1$ the limit of the iteraes T^m is exactly the operator $P(f; x) := (1-x)f(0) + xf(1)$, and under the conditions $T(e_0) = e_0$, $T(e_2) = e_2$, $T^\infty(e_1) = e_2$ the limit of the iteraes T^m is exactly the operator $V(f; x) := (1-x^2)f(0) + x^2f(1)$.

Theorem 3 Suppose that $T : C[0, 1] \rightarrow C[0, 1]$ is a positive linear operator.

(a) If $T(e_0) = e_0$, $T(e_1) = e_1$, $T^\infty(e_2) = e_1$, then $T^\infty = P$ and

$$|P(f; x) - T^m(f; x)| \leq \frac{3}{4}\omega_2\left(f; \sqrt{|x - T^m(e_2; x)|}\right). \quad (6)$$

(b) If $T(e_0) = e_0$, $T(e_1) \leq e_1$, $T(e_2) = e_2$, $T^\infty(e_1) = e_2$, then $T^\infty = V$ and

$$|V(f; x) - T^m(f; x)| \leq 3\omega_2(f; |T^m(e_1; x) - x^2|) + 2\omega_1(f; |T^m(e_1; x) - x^2|) + \frac{3}{4}\omega_2\left(f; \sqrt{2|T^m(e_1; x) - x^2|}\right).$$

(c) If $T(e_0) = e_0$, $T(e_1) \geq e_1$, $T(e_2) = e_2$, $T^\infty(e_1) = e_2$, then $T^\infty = V$ and

$$|V(f; x) - T^m(f; x)| \leq 3\omega_2(f; |T^m(e_1; x) - x^2|) + 2\omega_1(f; |T^m(e_1; x) - x^2|).$$

Remark 4 (i) Results similar to that of Theorem 3 (a) without the estimation (6) was obtained in [15] and [6]. (ii) Theorem 3 (b) and 3 (c) are new. They cover positive linear operators which preserve e_0 and e_2 . Convergence of overiterates of Bernstein operators and discrete type positive linear operators preserving e_0 and e_2 is studied in [3], [10].

Corollary 5 Let $T : C[0, 1] \rightarrow C[0, 1]$ be a positive linear operator such that

$$T(e_0) = e_0, \quad T(e_1) = e_1, \quad T(e_2) \leq ae_2 + be_1, a, b \in R \setminus \{0\}, \quad a + b = 1. \quad (7)$$

Then the pointwise approximation

$$|T^m(f; x) - P(f; x)| \leq \frac{3}{4}\omega_2\left(f; \sqrt{a^m x(1-x)}\right) \quad (8)$$

holds true for all $x \in [0, 1]$ and $f \in C[0, 1]$.

Remark 6 It is worth mentioning that the conditions

$$T(e_0) = e_0, \quad T(e_1) = e_1, \quad T(e_2) = ae_2 + be_1, a, b \in R \setminus \{0\}, \quad a + b = 1.$$

are satisfied by the many classical positive linear operators defined on $C[0, 1]$ and convergence of overiterates under these conditions was studied in [7], [15], [6]. The conditions (7) cover the classical MKZ and q-MKZ operators. The problem of convergence of overiterates of MKZ operators without quantitative estimate was studied in [5]. In Corollary 5 we give quantitative estimate for convergence.

Corollary 7 Suppose that $T : C[0, 1] \rightarrow C[0, 1]$ is a positive linear operator.

(a) If $T(e_0) = e_0$, $T(e_1) \leq e_1$, $T^\infty(e_1) = T^\infty(e_2) = 0$, then $T^\infty f = f(0)$ and

$$|f(0) - T^m(f; x)| \leq 3\omega_2(f; |T^m(e_1; x)|) + 2\omega_1(f; |T^m(e_1; x)|) + \frac{3}{4}\omega_2\left(f; \sqrt{|T^m(e_2; x)| + 2|T^m(e_1; x)|}\right).$$

(b) If $T(e_0) = e_0$, $T(e_1) \geq e_1$, $T^\infty(e_1) = T^\infty(e_2) = 1$, then $T^\infty f = f(1)$ and

$$|f(1) - T^m(f; x)| \leq 3\omega_2(f; |T^m(e_1; x) - 1|) + 2\omega_1(f; |T^m(e_1; x) - 1|) + \frac{3}{4}\omega_2\left(f; \sqrt{|T^m(e_2; x) - 1|}\right).$$

Remark 8 Corollary 7 covers the Bernstein-Stancu operators $S_n^{(\alpha, \beta, \gamma)}$ for some values of α, β, γ , see [9].

2 Proofs of the main results

Proof of Theorem 1. (b) For every convex $g \in C^2[0, 1]$, we have

$$g(t) \geq g(x) + g'(x)(t - x). \quad (9)$$

It follows that for any nondecreasing convex $g \in C^2[0, 1]$

$$\begin{aligned} T(g; x) &\geq g(x) + g'(x)(T(e_1; x) - x) \geq g(x), \\ g(x) &\leq T^m(g; x) \leq T^{m+1}(g; x) \leq \|g\|. \end{aligned}$$

In other words the sequence continuous functions $\{T^m(g; \cdot)\}$ is nondecreasing for any nondecreasing convex function $g \in C^2[0, 1]$. By Dini's theorem there exists $T^\infty(g; \cdot)$ such that $T^m(g; \cdot) \rightarrow T^\infty(g; \cdot)$ uniformly on $[0, 1]$, for any nonincreasing convex function $g \in C^2[0, 1]$. In particular,

$$\begin{aligned} \lim_{m \rightarrow \infty} \|T^m(e_1) - T^\infty(e_1)\| &= 0, \\ \lim_{m \rightarrow \infty} \|T^m(e_2) - T^\infty(e_2)\| &= 0. \end{aligned}$$

Let $g \in C^2[0, 1]$ be arbitrary. Introduce the following auxiliary functions

$$g_\pm(t) = \frac{1}{2} \|g''\| t^2 + \|g'\| t \pm g(t).$$

It is clear that

$$g'_\pm(t) = \|g''\| t + \|g'\| \pm g'(t) \geq 0, \quad g''_\pm(t) = \|g''\| \pm g''(t) \geq 0.$$

Therefore the functions $g_\pm(t)$ are nondecreasing convex for both choices of the sign. We have

$$\begin{aligned} 0 \leq T^{m+p}(g_\pm; x) - T^m(g_\pm; x) &= \frac{1}{2} \|g''\| (T^{m+p}(e_2; x) - T^m(e_2; x)) \\ &\quad + \|g'\| (T^m(e_1; x) - T^{m+p}(e_1; x)) \pm (T^{m+p}(g; x) - T^m(g; x)). \end{aligned}$$

It follows that

$$|T^{m+p}(g; x) - T^m(g; x)| \leq \frac{1}{2} \|g''\| |T^{m+p}(e_2; x) - T^m(e_2; x)| + \|g'\| |T^m(e_1; x) - T^{m+p}(e_1; x)|. \quad (10)$$

So $\{T^m(g; x)\}$ is a Cauchy sequence in $C[0, 1]$. Since $C[0, 1]$ is complete there is a function f^∞ such that

$$\lim_{m \rightarrow \infty} \|T^m(g) - f^\infty\| = 0.$$

Although this limit has been obtained for $g \in C^2[0, 1]$ only, it extends to all $f \in C[0, 1]$ by the Banach–Steinhaus theorem. Hence we find an operator $T^\infty : C[0, 1] \rightarrow C[0, 1]$ say, such that $T^\infty f := \lim_{m \rightarrow \infty} T^m(f) = f^\infty$, $f \in C[0, 1]$. Clearly, this operator is linear and positive.

Taking the limit as $p \rightarrow \infty$ in (10) we have

$$|(T^\infty - T^m)(g; x)| \leq \frac{1}{2} \|g''\| |(T^\infty - T^m)(e_2; x)| + \|g'\| |(T^m - T^\infty)(e_1; x)|.$$

Let $f \in C[0, 1]$. For $g \in C^2[0, 1]$ arbitrarily chosen we have the following estimate

$$\begin{aligned} |T^\infty(f; x) - T^m(f; x)| &\leq |(T^\infty - T^m)(f - g; x)| + |T^\infty(g; x) - T^m(g; x)| \\ &\leq 2 \|f - g\| + \frac{1}{2} \|g''\| |(T^\infty - T^m)(e_2; x)| + \|g'\| |(T^m - T^\infty)(e_1; x)| \\ &= 2 \|f - g\| + \|g'\| |(T^m - T^\infty)(e_1; x)| + \frac{1}{2} \|g''\| |(T^\infty - T^m)(e_2; x)|. \end{aligned} \quad (11)$$

We substitute now $g := B_n(Z_h f) \in C^2[0, 1]$, where $Z_h f$ is Zhuk's function. In (11) using the inequalities (1) and letting $\varepsilon \rightarrow 0$ we arrive at

$$\begin{aligned} |T^\infty(f; x) - T^m(f; x)| &\leq \frac{3}{2}\omega_2(f; h) + \frac{1}{h} \left(2\omega_1(f; h) + \frac{3}{2}\omega_2(f; h) \right) |(T^m - T^\infty)(e_1; x)| \\ &\quad + \frac{1}{\delta^2} \frac{3}{4}\omega_2(f; \delta) |(T^\infty - T^m)(e_2; x)| \end{aligned}$$

with $h > 0$ and $\delta > 0$. If

$$|(T^m - T^\infty)(e_1; x)| > 0, \quad |(T^m - T^\infty)(e_2; x)| > 0$$

taking $h = |(T^m - T^\infty)(e_1; x)|$ and $\delta^2 = |(T^\infty - T^m)(e_2; x)|$ yields the desired result. If $|(T^m - T^\infty)(e_1; x)| = 0$ and $|(T^\infty - T^m)(e_2; x)| > 0$, then

$$|T^\infty(f; x) - T^m(f; x)| \leq \frac{3}{2}\omega_2(f; h) + \frac{1}{\delta^2} \frac{3}{4}\omega_2(f; \delta) |(T^m - T^\infty)(e_2; x)|$$

for all $h > 0$. Taking $\delta = |(T^\infty - T^m)(e_2; x)|$, $h \rightarrow 0$ yields the desired result. If $|(T^m - T^\infty)(e_1; x)| > 0$ and $|(T^\infty - T^m)(e_2; x)| = 0$, then

$$|T^\infty(f; x) - T^m(f; x)| \leq \frac{3}{2}\omega_2(f; h) + \frac{1}{h} \left(2\omega_1(f; h) + \frac{3}{2}\omega_2(f; h) \right) |(T^m - T^\infty)(e_1; x)|$$

for all $h > 0$. Taking $h = |(T^\infty - T^m)(e_1; x)|$, yields the desired result. If $(T^m - T^\infty)(e_1; x) = 0$ and $(T^m - T^\infty)(e_2; x) = 0$, then

$$|T^\infty(f; x) - T^m(f; x)| \leq \frac{3}{2}\omega_2(f; h).$$

For $h \rightarrow 0$ we obtain $T^\infty(f; x) = T^m(f; x)$ for all $0 \leq x \leq 1$.

(a) It follows that from (9) that any nonincreasing convex $g \in C^2[0, 1]$

$$\begin{aligned} T(g; x) &\geq g(x) + g'(x)(T(e_1; x) - x) \geq g(x), \\ g(x) &\leq T^m(g; x) \leq T^{m+1}(g; x) \leq \|g\|. \end{aligned}$$

In other words the sequence continuous functions $\{T^m(g; \cdot)\}$ is nondecreasing for any nonincreasing convex function $g \in C^2[0, 1]$. By Dini's theorem there exists $T^\infty(g; \cdot)$ such that $T^m(g; \cdot) \rightarrow T^\infty(g; \cdot)$ uniformly on $[0, 1]$, for any nonincreasing convex function $g \in C^2[0, 1]$. In particular,

$$\begin{aligned} \lim_{m \rightarrow \infty} \|T^m(-e_1) - T^\infty(-e_1)\| &= 0, \\ \lim_{m \rightarrow \infty} \left\| T^m\left((e_0 - e_1)^2\right) - T^\infty\left((e_0 - e_1)^2\right) \right\| &= 0, \end{aligned}$$

and $\lim_{m \rightarrow \infty} \|T^m(e_2) - T^\infty(e_2)\| = 0$ since $T^m\left((e_0 - e_1)^2\right) = T^m(e_0) - 2T^m(e_1) + T^m(e_2)$

Let $g \in C^2[0, 1]$ be arbitrary. Introduce the following auxiliary functions

$$g_\pm(t) = \frac{1}{2} \|g''\| (1-t)^2 + \|g'\| (1-t) \pm g(t).$$

It is clear that

$$g'_\pm(t) = -\|g''\| (1-t) - \|g'\| \pm g(t) \leq 0, \quad g''_\pm(t) = \|g''\| \pm g(t) \geq 0.$$

Therefore the functions $g_\pm(t)$ are nonincreasing convex for both choices of the sign. We have

$$\begin{aligned} 0 \leq T^{m+p}(g_\pm; x) - T^m(g_\pm; x) &= \frac{1}{2} \|g''\| (T^{m+p}(e_2; x) - T^m(e_2; x)) \\ &\quad + (\|g''\| + \|g'\|) (T^m(e_1; x) - T^{m+p}(e_1; x)) \pm (T^{m+p}(g; x) - T^m(g; x)). \end{aligned}$$

The rest of the proof is similar to that of part (b). ■

Proof of Theorem 3. (a) It remains to show that $T^\infty(f) = P(f)$ for all $f \in C[0, 1]$. It is clear that it is enough to show this equality in $C^2[0, 1]$. Let $g \in C^2[0, 1]$. Define the following auxiliary functions.

$$G(x) := g(x) - P(g; x), \quad l := \frac{1}{2} \|G''\| = \frac{1}{2} \|g''\|, \quad g_\pm(x) := -lx^2 + lx \pm G(x).$$

It is clear that g_\pm is concave and nonnegative, since

$$g''_\pm(x) = -\|G''\| \pm G''(x) \leq 0, \quad G(0) = G(1) = 0.$$

It follows that

$$-l(x - x^2) \leq G(x) \leq l(x - x^2), \quad 0 \leq x \leq 1.$$

Applying the positive operator T^∞ we get

$$-l(T^\infty(e_1; x) - T^\infty(e_2; x)) \leq T^\infty(G; x) = T^\infty(g; x) - P(g; x) \leq l(T^\infty(e_1; x) - T^\infty(e_2; x)),$$

for all $0 \leq x \leq 1$, and consequently

$$T^\infty(g) = P(g)$$

for all $g \in C^2[0, 1]$, which completes the proof.

(b) The operator T^∞ of Theorem 2 satisfies

$$T^\infty(e_0) = e_0, \quad T^\infty(e_1) = e_2, \quad T^\infty(e_2) = e_2.$$

It remains to show that $T^\infty(f) = V(f)$ for all $f \in C[0, 1]$. It is clear that it is enough to show this equality in $C^2[0, 1]$.

Let $g \in C^2[0, 1]$. Define the following auxiliary functions.

$$\begin{aligned} G(x) &:= g(x) - V(g; x) = g(x) - (1 - x^2)g(0) - x^2g(1), \\ l &:= \frac{1}{2} \|g'' + 2g(0) - 2g(1)\|, \quad G''(x) = g''(x) + 2g(0) - 2g(1), \\ g_\pm(x) &= -lx^2 + lx \pm G(x). \end{aligned}$$

It is clear that g_\pm is concave and nonnegative, since

$$g''_\pm(x) = -\|G''\| \pm G''(x) \leq 0, \quad G(0) = G(1) = 0.$$

It follows that

$$-l(x - x^2) \leq G(x) \leq l(x - x^2), \quad 0 \leq x \leq 1.$$

Application of the positive operator T^∞ gives

$$-l(T^\infty(e_1; x) - T^\infty(e_2; x)) \leq T^\infty(G; x) = T^\infty(g; x) - V(g; x) \leq l(T^\infty(e_1; x) - T^\infty(e_2; x)), \quad 0 \leq x \leq 1,$$

and $T^\infty(g) = V(g)$ for all $g \in C^2[0, 1]$. ■

Proof of Corollary 5. By the induction we have

$$x^2 \leq T^m(e_2; x) \leq a^m x^2 + b(1 + a + \dots + a^{m-1})x = a^m x^2 + (1 - a^m)x.$$

So

$$0 \leq x - T^m(e_2; x) \leq a^m x(1 - x).$$

■

Proof of Corollary 7. The proof is based on the Taylor formula about the point 0 for part (a) and about the point 1 for part (b). ■

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